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Algebraic nature of shape-invariant and self-similar potentials

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Abstract. Self-similar potentials generalize the concept of shape invariance which was originally introduced to explore exactly solvable potentials in quantum mechanics. In this paper it is shown that the previously introduced algebraic approach to the latter can be generalized to the former. The infinite Lie algebras introduced in this context are shown to be closely related to the q-algebras. The associated coherent states are investigated.

1. Introduction

Supersymmetric quantum mechanics has been shown to be a useful technique to explore exactly solvable problems in quantum mechanics [1]. Introducing the function

$$W(x) \equiv -\frac{\hbar}{\sqrt{2m}}\left[\frac{\Psi_0'(x)}{\Psi_0(x)}\right] \tag{1.1}$$

where $\Psi_0(x)$ is the ground state wavefunction of the Hamiltonian $\hat{H}$, and the operators

$$\hat{A} \equiv W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.2}$$

$$\hat{A}^\dagger \equiv W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.3}$$

we can show that

$$\hat{A}|\Psi_0\rangle = 0 \tag{1.4}$$

and

$$\hat{H} - E_0 = \hat{A}^\dagger\hat{A}. \tag{1.5}$$

An integrability condition called shape invariance was introduced by Gendenshtein [2] and was cast into an algebraic form by Balantekin [3]. The shape-invariance condition can be written as

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{1.6}$$

where $a_{1,2}$ are a set of parameters. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of $\hat{x}$ and $\hat{p}$. Not all exactly solvable potentials are shape-invariant [4]. In the cases studied so far the parameters a_1 and a_2 are either related by a translation [4, 5]

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or a scaling [6]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{1.7}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{1.8}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{1.9}$$

the Hamiltonian takes the form

$$\hat{H} - E_0 = \hat{B}_+\hat{B}_-. \tag{1.10}$$

The Lie algebra associated by the shape invariance is defined with the commutation relations

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{1.11}$$

and

$$[\hat{B}_+, R(a_0)] = [R(a_1) - R(a_0)]\hat{B}_+ \tag{1.12}$$

$$[\hat{B}_+, \{R(a_1) - R(a_0)\}\hat{B}_+] = \{[R(a_2) - R(a_1)] - [R(a_1) - R(a_0)]\}\hat{B}^2 \tag{1.13}$$

and the Hermitian conjugates of the relations given in equations (1.12) and (1.13). In general there is an infinite number of such commutation relations, hence the appropriate Lie algebra is infinite-dimensional. In some special cases where the parameters are related by translation it is possible to reduce this infinite-dimensional algebra to a finite-dimensional one [3, 7, 8]. In this paper we explore the relationship between q-algebras and the cases where the parameters are related by scaling.

2. Coherent states

Since the operator $\hat{B}_-$ satisfies the relation

$$\hat{B}_-|\Psi_0\rangle = 0 \tag{2.1}$$

and the excited states can be written in the form

$$|\Psi_n\rangle \propto \hat{B}_+^n|\Psi_0\rangle \tag{2.2}$$

the operator $\hat{B}_-$ does not have a left inverse and the operator $\hat{B}_+$ does not have a right inverse. However, a right inverse for $\hat{B}_-$

$$\hat{B}_-\hat{B}_-^{-1} = 1 \tag{2.3}$$

and a left inverse for $\hat{B}_+$

$$\hat{B}_+^{-1}\hat{B}_+ = 1 \tag{2.4}$$

can be defined. Similarly in the Hilbert space of the eigenstates of the Hamiltonian, the inverse of $\hat{H}$ does not exist, but

$$\hat{H}^{-1}\hat{B}_+ = \hat{B}_-^{-1} \tag{2.5}$$

does. Also introducing

$$\hat{Q}^\dagger = \hat{H}^{-1/2}\hat{B}_+ \tag{2.6}$$

and its Hermitian conjugate

$$\hat{Q} = (\hat{Q}^\dagger)^\dagger = \hat{B}_-\hat{H}^{-1/2} \tag{2.7}$$

one can show that

$$\hat{Q}\hat{Q}^{\dagger} = \hat{1}. \tag{2.8}$$

The normalized excited states can then be written as

$$|\Psi_n\rangle = (\hat{Q}_+)^n|\Psi_0\rangle \tag{2.9}$$

provided that the ground state is normalized, i.e. $\langle\Psi_0|\Psi_0\rangle = 1$.

We introduce the coherent state for a shape-invariant potential as

$$\begin{aligned} |z\rangle &= |0\rangle + z\hat{B}_-^{-1}|0\rangle + z^2\hat{B}_-^{-2}|0\rangle + \cdots \\ &= \frac{1}{1 - z\hat{B}_-^{-1}}|0\rangle \end{aligned} \tag{2.10}$$

where we used the shorthand notation $|0\rangle \equiv |\Psi_0\rangle$. One can easily show that this state in an eigenstate of the operator $\hat{B}_-$:

$$\hat{B}_-|z\rangle = z|z\rangle \tag{2.11}$$

and satisfies the condition

$$(\hat{B}_- - z)\frac{\partial}{\partial z}|z\rangle = |z\rangle. \tag{2.12}$$

The state $|z\rangle$ coincides with the coherent state defined in [9] using a generalized exponential function. When the Lie algebra associated with the shape-invariant potential is $SU(1,1)$ [3,7], this is not the standard coherent state introduced in [10], but the state introduced by Barut and Girardello [11].

If a forced harmonic oscillator is in the ground state for $t = 0$, it evolves into the harmonic-oscillator coherent state. We must emphasize that the coherent states described here, in general, do not have such a simple dynamical interpretation. To illustrate this point we consider the time-dependent Hamiltonian

$$\hat{h}(t) = \hat{B}_+\hat{B}_- + f(t)[\mathrm{e}^{\mathrm{i}R(a_1)t/\hbar}\hat{B}_+ + \hat{B}_-\mathrm{e}^{-\mathrm{i}R(a_1)t/\hbar}] \tag{2.13}$$

where $f(t)$ is an arbitrary function of time. The solution of the time-evolution equation

$$\mathrm{i}\hbar\frac{\partial\hat{u}(t)}{\partial t} = \hat{h}(t)\hat{u}(t) \tag{2.14}$$

can be written as

$$\hat{u}(t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{B}_+\hat{B}_- t\right\}\hat{u}_I(t). \tag{2.15}$$

Substituting equation (2.14) into (2.15) one can show that $\hat{u}_I(t)$ satisfies the equation

$$\mathrm{i}\hbar\frac{\partial\hat{u}_I(t)}{\partial t} = f(t)[\hat{B}_+ + \hat{B}_-]\hat{u}_I(t). \tag{2.16}$$

The solution of equation (2.16) can be immediately written as

$$\hat{u}_I(t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t f(t')\,\mathrm{d}t'\,[\hat{B}_+ + \hat{B}_-]\right\}. \tag{2.17}$$

Hence under the time-evolution the ground state evolves into the state

$$|\Psi, t\rangle = \hat{u}_I(t)|0\rangle \tag{2.18}$$

which is not equivalent to the state given in equation (2.10).

3. Self-similar potentials and q-algebras

Shabat [12] and Spiridonov [13] discussed reflectionless potentials with an infinite number of bound states. These self-similar potentials are shown to be shape invariant in [6] and [14]. In this case the parameters are related by a scaling:

$$a_n = q^{n-1} a_1. \tag{3.1}$$

Barclay *et al* studied such shape-invariant potentials in detail [6]. In the simplest case studied by them the remainder of equation (1.6) is given by

$$R(a_1) = ca_1 \tag{3.2}$$

where c is a constant and the operator introduced in equation (1.7) by

$$\hat{T}(a_1) = \exp\left\{(\log q) a_1 \frac{\partial}{\partial a_1}\right\}. \tag{3.3}$$

Hence the energy eigenvalue of the nth excited state is

$$\begin{aligned} E_n &= R(a_1) + R(a_2) + \cdots + R(a_n) \\ &= (1 + q + q^2 + \cdots + q^{n-1}) ca_1 \\ &= \frac{1 - q^n}{1 - q} ca_1 \end{aligned} \tag{3.4}$$

which is the spectra of quantum oscillator [15]. Introducing the scaled operators

$$\hat{K}_\pm = \sqrt{q}\,\hat{B}_\pm \tag{3.5}$$

one can show that the commutation relations of equations (1.11)–(1.13) take the form

$$[\hat{K}_-, \hat{K}_+] = R(a_1) \tag{3.6}$$

and

$$[\hat{K}_+, R(a_1)] = (q - 1) R(a_1) \hat{K}_+. \tag{3.7}$$

Note that the algebra associated with the self-similar potentials is not a finite Lie algebra as $\hat{K}_+$ does not commute with $R(a_1)\hat{K}_+^n$:

$$[\hat{K}_+, (q-1)^n R(a_1) \hat{K}_+^n] = (q-1)^{n+1} R(a_1) \hat{K}_+^{n+1}. \tag{3.8}$$

Further introducing the operators

$$\hat{S}_+ = \hat{K}_+ R(a_1)^{-1/2} \tag{3.9}$$

and

$$\hat{S}_- = (\hat{S}_+)^\dagger = R(a_1)^{-1/2} \hat{K}_- \tag{3.10}$$

using equation (3.6) one can show that the standard q-deformed oscillator relation is satisfied:

$$\hat{S}_- \hat{S}_+ - q \hat{S}_+ \hat{S}_- = 1. \tag{3.11}$$

In the most general case for a self-similar potential the function $W(x)$ of equation (1.1) satisfies the condition [12, 13]

$$W(x) \xrightarrow{a_1 \to a_2} \sqrt{q}\, W(\sqrt{q}\, x) \tag{3.12}$$

or equivalently

$$\hat{A}^\dagger(x), \hat{A}(x) \xrightarrow{a_1 \to a_2} \sqrt{q}\, \hat{A}^\dagger(\sqrt{q}\, x), \sqrt{q}\, \hat{A}(\sqrt{q}\, x). \tag{3.13}$$

Inserting equation (3.13) into (1.6) one obtains the q-deformed form of equation (1.6):

$$\hat{A}(x) \hat{A}^\dagger(x) - q \hat{A}^\dagger(\sqrt{q}\, x) \hat{A}(\sqrt{q}\, x) = R(a_1). \tag{3.14}$$

Introducing the operators [16]

$$\hat{C} = \hat{A}(x)\mathrm{e}^{-\frac{1}{2}px\frac{\mathrm{d}}{\mathrm{d}x}} \tag{3.15}$$

and

$$\hat{C} = \mathrm{e}^{+\frac{1}{2}px\frac{\mathrm{d}}{\mathrm{d}x}}\hat{A}^{\dagger}(x) \tag{3.16}$$

where $q = \mathrm{e}^{p}$, equation (3.14) can be rewritten as

$$\hat{C}\hat{C}^{\dagger} - q\hat{C}^{\dagger}\hat{C} = R(a_1). \tag{3.17}$$

Note that an algebraic approach to the self-similar potentials was already introduced in [7, 8]. Here we would like to establish that our algebra is identical to that in [8]. To this end we introduce

$$\hat{J}_3 = -\frac{1}{p}\log a_0. \tag{3.18}$$

Using equation (3.18), equation (1.11) can be written as

$$[\hat{B}_-, \hat{B}_+] = c\exp(-p\hat{J}_3). \tag{3.19}$$

Using equation (1.7), one can show that for an arbitrary function $f(a_n)$ of the parameters a_n we can write

$$f(a_n)\hat{B}_+ = \hat{B}_+ f(a_{n-1}) \tag{3.20}$$

and

$$f(a_n)\hat{B}_- = \hat{B}_- f(a_{n+1}). \tag{3.21}$$

Using equations (3.20) and (3.21) one can easily prove the commutation relation

$$[\hat{J}_3, \hat{B}_{\pm}] = \pm\hat{B}_{\pm}. \tag{3.22}$$

Equations (3.19) and (3.22) represent the algebra introduced in [8]. This algebra is a deformation of the standard $SO(2, 1)$ algebra.

The coherent state is easy to construct. The term multiplying z^n in equation (2.10) is

$$\begin{aligned} z^n\hat{B}_-^{-n}|0\rangle &= z^n(\hat{H}^{-1}\hat{B}_+)^n|0\rangle \\ &= [E_n(E_n - E_{n-1})(E_n - E_{n-2})\ldots(E_n - E_1)]^{-1/2}|n\rangle \end{aligned} \tag{3.23}$$

where $|n\rangle$ is the shorthand notation for the nth excited state $|\Psi_n\rangle$ the energy of which is E_n. Inserting equation (3.4) into (3.23) one can write down the coherent state as

$$|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[R(a_1)]^n}}\frac{(1-q)^{n/2}q^{-n(n-1)/4}}{\sqrt{(q;q)_n}}|n\rangle \tag{3.24}$$

where the q-shifted factorial $(q; q)_n$ is defined as $(z; q)_0 = 1$ and $(z; q)_n = \prod_{j=0}^{n-1}(1 - zq^j)$, $n = 1, 2, \ldots$. One observes that the norm of this state belongs to the one-parameter family of q-exponential functions considered by Floreanini *et al* [17]. Alternative approaches to the coherent states for the q-algebras were given in [15] and [18] and were used to construct path integrals in [19]. In addition Fukui has shown that a coherent state associated with shape invariance leads to a particular q-coherent state [14].

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